Massive Schwinger model and its confining aspects on curved space–time

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Abstract

Using a covariant method to regularize the composite operators, we obtain the bosonized action of the massive Schwinger model on a classical curved background. Using the solution of the bosonic effective action, the energy of two static external charges with finite and large distance separation on a static curved space–time is obtained. The confining behavior of this model is also explicitly discussed.

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1 Introduction

Two dimensional quantum electrodynamics or the Schwinger model [1] may be served as a laboratory for studying four dimensional gauge theories and important phenomena such as confinement, screening, chiral symmetries and $\cdots$. On a flat space–time, it is known that in the massless Schwinger model via a peculiar two dimensional Higgs phenomenon, the photon becomes massive and the Coulomb force is replaced by a finite range force, giving rise to the screening phase. When the dynamical fermions are massive, the model is not exactly soluble, but a semiclassical analysis reveals a linear rise energy for opposites test charges q and $\bar{q}$, binding them into $q\bar{q}$ pairs [2]. By taking into accounts the finite distance corrections, one can show that besides the linear confining term, the potential is also composed of a screening term which is modified with respect to the massless case [3].

One of the interesting questions is how a curved background can modify these effects? We think that this is an important question, because it can be viewed as a first step in studying these physical effects in the context of quantum gravity. Moreover, they may have applications in string theory and quantum gravity coupled to nonconformal matter (note that the kinetic term of the gauge fields spoils the conformal invariance of the theory).

Although the Schwinger model on curved space–time has been studied in several papers but the confining aspects of the model is still unclear (for a discussion about the subtleties in determining the confining phase of the Schwinger model on curved space see [4]). For example the author of [5] has suggested that the curvature of the space does not change the confining behavior of the massive Schwinger model, and in [6] it has been claimed that on a particular black hole, the massless Schwinger model remains in screening phase. Also in [4], by comparison the role of temperature and the curvature, it has been argued that the curvature may modify the confining or screening nature of the Schwinger model. More recently, this model has been investigated on a constant negative curvature space–time (i.e., the Poincare half–plane) [7], and it has been found that the confining feature of the Schwinger model depends on the geometry of the space–time, for example, in different regions of Poincare half–plane the system is in different phases (i.e. confining or screening phases), which is in contradiction with flat space–time results. Also in [8], the confining aspects of the massless Schwinger model on de Sitter space in different coordinates has been discussed and the results have been extended to non–abelian situation.

In this paper we want to study the massive Schwinger model on a general noncompact Riemann surface, and try to obtain, as much as possible, information about the confinement in this background. We consider only the effects of geometry on confinement phenomenon and ignore nontrivial topologies.

In the Schwinger model the potential of external charges can be obtained using several different methods, for example : (i) Integrating over the fermionic fields (or equivalently

over bosonic fields in the bosonized version of the model), results an effective action for the gauge fields, from which the potential can be extracted by solving the corresponding equations of motion. (ii) Integrating over gauge fields in the bosonized action and then obtaining the static solutions of the equation of motion and deducing the potential of the system as the difference between the Hamiltonian in the presence and absence of external charges respectively. In this paper we follow the second procedure.

The plan of the paper is as follows: In section 2, our main task is to obtain the bosonic version of the massive Schwinger model on a general non–compact curved background. To find this bosonic representation, it is necessary to employ an appropriate normal ordering description in defining the composite fields, which are always present in the bosonic representations. As we will see, this needs several change of field variables in order to reduce our main model (interacting massive fermionic fields in curved background) to a free massless fermionic field theory in flat space–time. Then using the equivalence of this theory with a massless bosonic field theory, we may obtain the appropriate normal ordering description, from which our bosonic representation can be found.

In section 3, by introducing two external charges q and $\bar{q}$ in the bosonized action, and by restricting ourselves to static metrics (so that the static potential between probe charges becomes meaningful), we obtain the energy of widely separated external charges for small dynamical fermions mass. Obviously, the final result depends on the metric. For a specific example (the full de Sitter space–time), we will show that one recovers a confining behavior in the above mentioned limit. We also show that when $e'/q \in Z$, where e' (q) is the charge of external (dynamical) charges, the external charges can not modify the energy .This is the same result as the flat case. Note that in section 2 where we use the path integral approach, it is more convenient to consider the Euclidean signature, but in section 3, where we are going to calculate the energy, it is better to work with Minkowskian signature, in which the concept of energy is more natural.

To study the same problem but with external charges with finite distance separation, in section 4 we restrict ourselves to the case in which the gauge coupling is very large with respect to the variation of the metric. The main reason for such a choice is that, the massive scalar Green function, which is needed in calculating the energy, is only known for a few number of space–times. But if we restrict ourselves to the above mentioned metrics, we can use the WKB approximation to find a general expression for the energy of external charges. Like the flat case, the energy takes a screening correction terms beside the confining term. As a result we show that at $m = 0$, the phase structure of the system depends on the metric and in contrast to the flat case, the model can be in confining phase in specific cases (such as the full de Sitter space–time) which is in agreement with the result of [8].

2 Bosonization of the massive Schwinger model

All two dimensional spaces are conformally flat, hence any noncompact Riemann surface is described by the metric

$$ds^2 = \sqrt{g}(dt^2 + dx^2), \tag{1}$$

where $\sqrt{g}$ is the conformal factor. On this space–time, the massive Schwinger model is described by the partition function

$$Z = \int DA_\mu D\psi D\psi^\dagger \exp\{-\int d^2x[\sqrt{g}(\psi^\dagger\gamma^\mu(\partial_\mu - iqA_\mu)\psi + m\psi^\dagger\psi) + \frac{1}{2\sqrt{g}}F^2]\}, \tag{2}$$

where $\gamma^\mu \equiv \hat{\gamma}^a e_a^\mu$ are the curved space counterparts of Hermitian Dirac gamma matrices: $\hat{\gamma}^0 = \sigma_2$; $\hat{\gamma}^1 = \sigma_1$. σ_i are Pauli matrices. ∂_μ is the covariant derivative and the zweibeins are defined through

$$g_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}, \qquad g^{\mu\nu} = e_a^\mu e_b^\nu \delta^{ab}. \tag{3}$$

The metric components are $g^{\mu\nu} = (1/\sqrt{g})\hat{\delta}^{\mu\nu}$, where $\hat{\delta}^{\mu\nu} = \hat{\delta}_{\mu\nu} = diag(1,1)$. q and m are the charge and the mass of dynamical fermions respectively. In eq.(2), the dimension of q and m is the inverse of the length dimension and e is dimensionless. The (dual) field strength F, is described through $F = \hat{\epsilon}^{\mu\nu}\partial_\mu A_\nu$ in terms of gauge fields A_μ, where $\hat{\epsilon}^{\mu\nu} = \hat{\epsilon}_{\mu\nu}$ and $\hat{\epsilon}^{01} = -\hat{\epsilon}_{10} = 1$. The partition function (2) can be written as

$$Z = \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} < \prod_{j=1}^{k} \int d^2x_j \sqrt{g(\mathbf{x_j})}\psi^\dagger(\mathbf{x}_j)\psi(\mathbf{x}_j) >, \tag{4}$$

where $\mathbf{x} = (x,t)$. In eq.(4), the expectation values are computed from the Lagrangian

$$L = \sqrt{g}\psi^\dagger\gamma^\mu(\partial_\mu - iqA_\mu)\psi + \frac{1}{2\sqrt{g}}F^2. \tag{5}$$

In terms of the new fermionic variables $\tilde{\psi}$ and $\tilde{\psi}^\dagger$:

$$\tilde{\psi} = \exp(q\gamma_5\varphi)\psi,$$

$$\tilde{\psi}^\dagger = \psi^\dagger \exp(q\gamma_5\varphi), \tag{6}$$

where $A_\mu = \hat{\epsilon}_{\mu\nu}\partial_\nu\varphi$ and $\gamma_5 = -i\hat{\gamma}^1\hat{\gamma}^0$, the Lagrangian (5) can be rewritten as [9]

$$L = \sqrt{g}\tilde{\psi}^\dagger\gamma^\mu\partial_\mu\tilde{\psi} + \sqrt{g}(-\frac{\mu^2}{2}\varphi\Delta\varphi + \frac{1}{2}\varphi\Delta\Delta\varphi), \tag{7}$$

in which Δ is the Laplace–Beltrami operator: $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ and

$$\mu = \frac{q}{\sqrt{\pi}}. \tag{8}$$

In this way the massless part has been became a free field theory on a curved background with an effective action containing an anomalous term $(-(\mu^2/2)\sqrt{g}\varphi\triangle\varphi)$ coming from the Jacobian of the transformation (6) . The term $(1/2)\sqrt{g}\varphi\Delta(\Delta-\mu^2)\varphi$, is the effective Lagrangian density of the gauge fields. The fermionic part of the Lagrangian (7) is free and hence is invariant under Weyl transformation: $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$ and $\psi \rightarrow \Omega^{-\frac{1}{2}}\psi$. Choosing $\Omega = g^{-\frac{1}{4}}$, one obtains

$$\begin{gathered} g_{\mu\nu} \rightarrow \hat{\delta}_{\mu\nu}, \\ \tilde{\psi} \rightarrow \lambda = g^{\frac{1}{8}}\tilde{\psi}, \qquad \tilde{\psi}^\dagger \rightarrow \lambda^\dagger = g^{\frac{1}{8}}\tilde{\psi}^\dagger. \end{gathered} \tag{9}$$

Therefore the partition function (2) can be reduced to

$$Z = \sum_{k=0}^{\infty} \frac{(-m)^k}{k!} < \prod_{j=1}^{k} \int d^2x_j g^{\frac{1}{4}}(\mathbf{x}_j)\lambda^\dagger(\mathbf{x}_j)\exp[-2q\gamma_5\varphi(\mathbf{x}_j)]\lambda(\mathbf{x}_j) >, \tag{10}$$

where the expectation values are computed from

$$L = \lambda^\dagger\hat{\gamma}^a\partial_a\lambda + \frac{1}{2}\sqrt{g}\varphi\triangle(\triangle-\mu^2)\varphi. \tag{11}$$

Therefore the massless fermionic part of the theory is reduced to a free fermionic field theory on a flat space–time. Using $(\gamma_5)^{2n} = 1$, it can be easily shown that

$$\lambda^\dagger\exp(-2q\gamma_5\varphi)\lambda = \lambda^\dagger[\frac{1+\gamma_5}{2}\exp(-2q\varphi) + \frac{1-\gamma_5}{2}\exp(2q\varphi)]\lambda. \tag{12}$$

The fermionic part of (11) is chirally invariant, hence the only non–zero terms in (10) are those with equal number of σ_+ and σ_- [10], where $\sigma_\pm = \lambda^\dagger(1\pm\gamma_5)\lambda/2$. Hence

$$Z = \sum_{k=0}^{\infty} \frac{(-m)^{2k}}{(k!)^2} < \prod_{j=1}^{k} \int d^2x_j d^2y_j g^{\frac{1}{4}}(\mathbf{x}_j)g^{\frac{1}{4}}(\mathbf{y}_j)\exp[2q\varphi(\mathbf{y}_j)]\exp[-2q\varphi(\mathbf{x}_j)]\sigma_+(\mathbf{x}_j)\sigma_-(\mathbf{y}_j) > . \tag{13}$$

The expectation values of the fermionic part can be established in the same manner as in the flat case [11]:

$$\int D\lambda D\lambda^\dagger \prod_{j=1}^{k}\sigma_+(\mathbf{x}_j)\sigma_-(\mathbf{y}_j)\exp(-\int d^2x\lambda^\dagger\hat{\gamma}^a\partial_a\lambda) = (\frac{1}{2\pi})^{2k}\frac{\prod_{i>j}^{k}(\mathbf{x}_i-\mathbf{x}_j)^2(\mathbf{y}_i-\mathbf{y}_j)^2}{\prod_{i,j=1}^{k}(\mathbf{x}_i-\mathbf{y}_j)^2}. \tag{14}$$

But the Green function of the massless scalar field $\phi(\mathbf{x})$ is $<\phi(\mathbf{x})\phi(\mathbf{y})>= -1/(4\pi)\ln[\tilde{\epsilon}^2(\mathbf{x}-\mathbf{y})^2]$, where the mass $\tilde{\epsilon}^2$ $(\tilde{\epsilon}^2 \rightarrow 0)$ is introduced to avoid infrared divergences. This is equivalent to infrared renormalization of exponential of massless scalar field discussed in [12] in operator language. So using

$$\int D\phi(\mathbf{x})\exp[\Sigma_{j=1}^{k}\frac{\alpha_j^2}{2}D(\mathbf{x}_j,\mathbf{x}_j)]\exp[\Sigma_{j=1}^{k}i\alpha_j\phi(\mathbf{x}_j)]\exp[-\frac{1}{2}\int(\partial_\mu\phi(\mathbf{x}))^2d^2x] =$$

$$\delta_{\sum_{j=1}^{k}\alpha_j}\exp[-\sum_{i<j}^{k}\alpha_i\alpha_j D(\mathbf{x}_i,\mathbf{x}_j)], \tag{15}$$

in which we have used $\tilde{\epsilon}^2 \to 0$ limit and $D(\mathbf{x}_i,\mathbf{x}_j) = -1/(4\pi)\ln(\mathbf{x}_i - \mathbf{x}_j)^2$, one can write the eq.(14) in the form

$$(\frac{1}{2\pi})^{2k}\exp[2\pi\Sigma_{i=1}^{k}D(\mathbf{x}_i,\mathbf{x}_i)]\exp[2\pi\Sigma_{i=1}^{k}D(\mathbf{y}_i,\mathbf{y}_i)]\times$$

$$\int D\phi(\mathbf{x})\exp\{2i\sqrt{\pi}[\sum_{j=1}^{k}(\phi(\mathbf{x}_j)-\phi(\mathbf{y}_j))]\}\exp[-\int d^2x\frac{1}{2}(\partial_\mu\phi(\mathbf{x}))^2]. \tag{16}$$

Hence the eq.(13) can be written as

$$\begin{aligned} Z \;=\; & \sum_{k=0}^{\infty}\frac{m^{2k}}{(2\pi)^{2k}(k!)^2} < \prod_{j=1}^{k}\int d^2x_j d^2y_j g^{\frac{1}{4}}(\mathbf{x}_j)g^{\frac{1}{4}}(\mathbf{y}_j)\exp[2\pi D(\mathbf{x}_j,\mathbf{x}_j)]\exp[2\pi D(\mathbf{y}_j,\mathbf{y}_j)]\times \\ & \exp[-2q\varphi(\mathbf{x}_j)]\exp[2q\varphi(\mathbf{y}_j)]\exp[2i\sqrt{\pi}\phi(\mathbf{x}_j)]\exp[-2i\sqrt{\pi}\phi(\mathbf{y}_j)] >, \end{aligned} \tag{17}$$

where the expectation values is calculated from the following Lagrangian (note that $\sqrt{g}g^{\mu\nu} = \hat{\delta}^{\mu\nu}$):

$$L = \frac{1}{2}\sqrt{g}\varphi\triangle(\triangle-\mu^2)\varphi + \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{18}$$

Now if we note that there is the charge conservation law in contracting the composite fields $e^{i\beta\phi}$ [10], then the partition function (17) can be written as

$$\begin{aligned} Z \;=\; & \sum_{k=0}^{\infty}\frac{m^{2k}}{(2\pi)^{2k}(k!)^2} < \prod_{j=1}^{k}\int d^2x_j d^2y_j h(\mathbf{x}_j)h(\mathbf{y}_j)\times \\ & \exp[2i\sqrt{\pi}(\phi+\frac{iq}{\sqrt{\pi}}\varphi)(\mathbf{x}_j)]\exp[-2i\sqrt{\pi}(\phi+\frac{iq}{\sqrt{\pi}}\varphi)(\mathbf{y}_j)] > \\ =\; & < \sum_{k=0}^{\infty}\frac{1}{k!}\{\frac{m}{\pi}\int d^2x h(\mathbf{x})\cos[2\sqrt{\pi}(\phi(\mathbf{x})+\frac{iq}{\sqrt{\pi}}\varphi(\mathbf{x})]\}^k > \\ =\; & < \exp\{\frac{m}{\pi}\int d^2x h(\mathbf{x})\cos[2\sqrt{\pi}(\phi(\mathbf{x})+\frac{iq}{\sqrt{\pi}}\varphi(\mathbf{x}))]\} >, \end{aligned} \tag{19}$$

in which

$$h(\mathbf{x}) = g^{\frac{1}{4}}(\mathbf{x})\exp[2\pi D(\mathbf{x},\mathbf{x})]. \tag{20}$$

By changing the field variable $\phi \to \phi - (iq/\sqrt{\pi})\varphi$, and using $A_\mu = \hat{\epsilon}_{\mu\nu}\partial_\nu\varphi$, the partition function (19) becomes (note that this change of field variable is somehow inverse of (6) and couples the matter and gauge fields)

$$\begin{aligned} Z \;=\; & \int DA_\mu D\phi\exp\{-\int d^2x\sqrt{g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - i\frac{q}{\sqrt{\pi}}\epsilon^{\mu\nu}A_\mu\partial_\nu\phi + \frac{1}{2g}F^2 \\ & -\frac{m}{\pi}g^{-\frac{1}{4}}(\mathbf{x})\exp[2\pi D(\mathbf{x},\mathbf{x})]\cos(2\sqrt{\pi}\phi(\mathbf{x})]\}, \end{aligned} \tag{21}$$

where $\epsilon^{\mu\nu} = (1/\sqrt{g})\hat{\epsilon}^{\mu\nu}$. By integrating over the gauge fields one can see that the effective Lagrangian of the field ϕ is

$$L_{\text{eff.}} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\mu^2}{2}\sqrt{g}\phi^2 - \frac{m}{\pi}g^{\frac{1}{4}}\exp[2\pi D(\mathbf{x},\mathbf{x})]\cos[2\sqrt{\pi}\phi(\mathbf{x})]. \quad (22)$$

Now we want to convert the mass term in eq.(22) to a more convenient form, by absorbing covariantly the ultraviolet divergence in the mass m or in operator language, by defining an appropriate normal ordering prescription (N_μ) with respect to the scale μ [15]. To do so in the flat case, we write $\exp[2\pi D(\mathbf{x},\mathbf{x})]$ as

$$\begin{aligned} \exp[2\pi D(\mathbf{x},\mathbf{x})] &= \exp[-2\pi(G_\mu(\mathbf{x},\mathbf{x}) - D(\mathbf{x},\mathbf{x}))]\exp[2\pi G_\mu(\mathbf{x},\mathbf{x})] \\ &= \exp(\gamma + \ln\frac{\mu}{2})\exp[2\pi G_\mu(\mathbf{x},\mathbf{x})], \end{aligned} \quad (23)$$

where G_μ is the Green function of a massive scalar field with mass μ; which for $\sqrt{g} = 1$ is $G_\mu^{\text{flat}} = 1/(2\pi)K_0(\mu|\mathbf{x}-\mathbf{y}|)$. K_0 is the modified Bessel function of the second kind and γ is the Euler constant. We can now absorb $\exp[2\pi G_\mu(\mathbf{x},\mathbf{x})]$ in redefinition (renormalization) of mass m:

$$m\psi^\dagger\psi = M\frac{q\exp\gamma}{2\pi^{\frac{3}{2}}}\cos(2\sqrt{\pi}\phi), \quad (24)$$

where $M \equiv m\exp[2\pi G_\mu^{\text{flat}}(\mathbf{x},\mathbf{x})]$. In operator language [10], this is equivalent to the well known result [15]:

$$m\psi^\dagger\psi(\mathbf{x}) = -m\frac{q\exp(\gamma)}{2\pi^{\frac{3}{2}}}N_\mu\cos[2\sqrt{\pi}\phi(\mathbf{x})]. \quad (25)$$

Now let us consider the normal ordering in curved background. In this case, one can use a covariant point–splitting method in order to regularize the composite operator ϕ^2 (our regularization method should respect general covariance)

$$< \phi^2_{\text{reg.}}(\mathbf{x}) >= \lim_{\mathbf{x}\to\mathbf{x}'}[G_\mu(\mathbf{x},\mathbf{x}') - G_\mu^{DS}(\mathbf{x},\mathbf{x}')], \quad (26)$$

where $G_\mu^{DS}(\mathbf{x},\mathbf{x}')$ (extracted from DeWitt–Schwinger expansion) is the counterterm needed to regularize the ultraviolet divergence of the Green function [13]

$$G_\mu^{DS}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}[2\gamma + \ln(\mu^2\epsilon^2) - \frac{1}{6}\frac{R(\mathbf{x})}{\mu^2} + O(\epsilon^2)]. \quad (27)$$

R is the scalar curvature of the space and ϵ is one half of the proper distance between $\mathbf{x}$ and $\mathbf{x}'$. (Note that in the flat case, $R = 0$, eq.(27) coincides with $\lim_{\mathbf{x}'\to\mathbf{x}} G_\mu^{\text{flat}}(\mathbf{x},\mathbf{x}')$, and from (26) we obtain the usual regularization which kills out the loops.) Now if we follow the same steps as in the flat case, but here using $G_\mu^{DS}(\mathbf{x},\mathbf{x})$ instead of $G_\mu^{\text{flat}}(\mathbf{x},\mathbf{x})$, we obtain

$$m\psi^\dagger\psi(\mathbf{x}) = -m\frac{1}{\pi g^{\frac{1}{4}}(\mathbf{x})}\exp\{-2\pi[G_\mu^{DS}(\mathbf{x},\mathbf{x}) - D(\mathbf{x},\mathbf{x})]\}\tilde{N}_\mu\cos[2\sqrt{\pi}\phi(\mathbf{x})], \quad (28)$$

in which we have defined the normal ordering $\tilde{N}_\mu$ as

$$\tilde{N}_\mu \cos[2\sqrt{\pi}\phi(\mathbf{x})] = \exp[2\pi G_\mu^{DS}(\mathbf{x},\mathbf{x})] \cos[2\sqrt{\pi}\phi(\mathbf{x})]. \tag{29}$$

In the static curved space–time we can use a normal ordering which like the flat case kills the loops (the vacuum is now defined using the global time–like killing vector of the static space–time [14]). In the same manner as the flat case, we obtain

$$m\psi^\dagger\psi = -\frac{m}{\pi g^{\frac{1}{4}}} \exp[2\pi(D - G_\mu)(\mathbf{x},\mathbf{x})] N_\mu \cos[2\sqrt{\pi}\phi(\mathbf{x})]. \tag{30}$$

It is easy to show that (30) is equal to (28). From (30) (or equivalently (28) for static space–time), we obtain

$$< \psi^\dagger\psi(\mathbf{x}) >_{m=0} = -\frac{1}{\pi g^{\frac{1}{4}}} \exp[-2\pi G(\mathbf{x},\mathbf{x})], \tag{31}$$

where $G(\mathbf{x},\mathbf{x}) \equiv G_\mu(\mathbf{x},\mathbf{x}) - D(\mathbf{x},\mathbf{x})$. So, as we expect, the expectation value $< \psi^\dagger\psi >$ is independent of the method of regularization (or normalization) of composite fields.

3 Quark– antiquark potential in the massive Schwinger model on static curved spaces: widely separated charges

In this section we obtain the energy of two external static charges introduced into the massive Schwinger model. We consider an infinite static conformally flat space-time, with trivial topology, described by the metric

$$ds^2 = \sqrt{g(x)}(dt^2 - dx^2). \tag{32}$$

The bosonized action of the massive Schwinger model, in the presence of the covariantly conserved external current

$$J^0(x) = \frac{e'}{\sqrt{g}}(-\delta(x-b) + \delta(x-a)), \qquad J^1 = 0, \tag{33}$$

describing two opposite point charges $-e'$ and e' located at $x = b$ and $x = a$ respectively, is

$$\begin{aligned} S &= \int d^2x[\frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{q}{\sqrt{\pi}}F\phi + \frac{m}{\pi}g^{\frac{1}{4}}\exp[-2\pi G(x,x)]N_\mu\cos(2\sqrt{\pi}\phi(\mathbf{x})) \\ &\quad + \frac{1}{2\sqrt{g}}F^2 + \tilde{\eta}F], \end{aligned} \tag{34}$$

where

$$\tilde{\eta}(x) = e'[\theta(x-a) - \theta(x-b)] = \begin{cases} e', & a < x < b \\ 0, & x \notin [a,b]. \end{cases} \tag{35}$$

Eq.(34) is the Minkowskian version of (21) in the presence of external current (33). In deriving (34) we have also used eq.(30). Integrating over the field F results

$$\begin{aligned} S_{\text{eff.}} &= \int d^2x[\frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m}{\pi}g^{\frac{1}{4}}\exp[-2\pi G(x,x)]N_\mu\cos(2\sqrt{\pi}\phi(\mathbf{x})) \\ &\quad -\frac{q^2\sqrt{g}}{2\pi}(\eta+\phi)^2], \end{aligned} \tag{36}$$

in which we have defined $\eta = (\sqrt{\pi}/q)\tilde{\eta}$. For widely separated charges, $\tilde{\eta}$ is equal to the constant e' in the whole space (in the next section we will obtain the necessary corrections for finite charge separation distance). If one changes the variable $\phi \to \phi - \eta$, finds

$$\begin{aligned} S_{\text{eff.}} &= \int d^2x[\frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m}{\pi}g^{\frac{1}{4}}\exp[-2\pi G(x,x)]N_\mu\cos[2\sqrt{\pi}(\phi-\eta)(\mathbf{x})] - \\ &\quad \frac{\mu^2}{2}\sqrt{g}\phi^2]. \end{aligned} \tag{37}$$

For $(e'/q) \in \mathrm{Z}$, the action is not modified by the presence of external charges, hence the energy is not changed. In other words, in this case, external probes with charge e' are screened by dynamical fermions. The same effect occurs in flat space–time [2].

Now let us restrict ourselves to $|\phi| \ll 1$ scalar fields. In this regime, the action (36) becomes Gaussian and the classical solutions of the action coincide with the quantum ones. The classical equation of motion for static field ϕ is

$$[\partial_1^2 - \mu^2\sqrt{g} - 4\sqrt{\pi}m\Sigma\sqrt{g}\cos(2\sqrt{\pi}\eta)]\phi = -2\sqrt{\pi}m\Sigma\sqrt{g}\sin(2\sqrt{\pi}\eta), \tag{38}$$

in which Σ is defined through

$$\Sigma(x) \equiv \frac{1}{g^{\frac{1}{4}}\pi}\exp[-2\pi G(x,x)]. \tag{39}$$

The solution of eq.(38) is

$$\phi = O^{-1}2\sqrt{\pi}m\Sigma\sqrt{g}\sin(2\sqrt{\pi}\eta), \tag{40}$$

where

$$O = -\partial_1^2 + \mu^2\sqrt{g} + 4m\pi\Sigma\sqrt{g}\cos(2\sqrt{\pi}\eta). \tag{41}$$

Now as the eq.(40) can be written as

$$\phi = [1 - O^{-1}(O - 2\sqrt{\pi}m\Sigma\sqrt{g})]\sin(2\sqrt{\pi}\eta), \tag{42}$$

therefore the condition $|\phi| \ll 1$ is satisfied when $m\Sigma \ll \mu^2$. The energy of the system is

$$E = \int T_0^0 dx = -\int L dx, \tag{43}$$

where T^{μ}_{ν} is the energy momentum tensor and E is the energy measured by a static observer with respect to the coordinate (32). By substituting (40) into (36), we obtain

$$E(\eta) = -\int_a^b [2\pi m^2 \sin^2(2\sqrt{\pi}\eta)\Sigma\sqrt{g}O^{-1}\Sigma\sqrt{g} + m\Sigma\sqrt{g}\cos(2\sqrt{\pi}\eta)]dx. \tag{44}$$

Up to the first order of m and by considering $m\Sigma \ll \mu^2$, one can arrive at

$$E(\eta) \cong -m\cos(2\sqrt{\pi}\eta)\int_a^b \Sigma(x)\sqrt{g}\,dx. \tag{45}$$

Therefore the energy of external charges is

$$\begin{aligned} E_{\text{ext.}} &\equiv E(\eta) - E(0) \\ &= m[1-\cos(2\pi\frac{e'}{q})]\int_a^b \Sigma(x)\sqrt{g}dx. \end{aligned} \tag{46}$$

This completes our general result for the energy of widely separated external charges $\pm e'$ in a static curved background. This result coincides exactly with the result of [8], in which the same problem has been discussed using the fermionic action. As is clear from (45), the determination of the energy is now complicated by the presence of the conformal factor ($\sqrt{g}$) and Σ, which the latter can be expressed in terms of Seeley DeWitt coefficients although not explicitly known for general curved space–times. To obtain an insight about this result, let us consider a specific example.

Example: complete de Sitter space–time

Consider the following geodesically complete space–time [16]

$$ds^2 = (1+\frac{x^2}{\lambda^2})dt^2 - \frac{dx^2}{1+x^2/\lambda^2}, \tag{47}$$

which represents the full two–dimensional de Sitter space–time, with scalar curvature $R = 2/\lambda^2$. By the change of coordinate $x \to x(r)$, where $dr^2 = dx^2/(1+x^2/\lambda^2)^2$ or $r = \lambda\cos^{-1}[\lambda/(x^2+\lambda^2)^{\frac{1}{2}}]$, this space takes the conformally flat form

$$\begin{aligned} ds^2 &= (1+\frac{x^2}{\lambda^2})(dt^2 - dr^2) \\ &= \frac{1}{\cos^2(\frac{r}{\lambda})}(dt^2 - dr^2), \quad -\frac{\pi}{2} < \frac{r}{\lambda} < \frac{\pi}{2}. \end{aligned} \tag{48}$$

On spaces with constant positive curvature R, $G_\mu(\mathbf{x},\mathbf{y})$ is [17]

$$\lim_{\mathbf{x}\to\mathbf{y}} G_\mu(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\{2\gamma + \ln(\frac{R\epsilon^2}{2}) + \Psi(\frac{1}{2}+\alpha) + \Psi(\frac{1}{2}-\alpha)\}, \tag{49}$$

where $\alpha^2 = 1/4 + 2\mu^2/R$ and Ψ is the digamma function. As a consequence Σ is

$$\Sigma = \frac{1}{\pi}\exp[\gamma + \frac{1}{2}\ln(\frac{R}{2}) + \frac{1}{2}\Psi(\frac{1}{2}+\alpha) + \frac{1}{2}\Psi(\frac{1}{2}-\alpha)]. \tag{50}$$

The energy of widely separated external charges is then

$$E_{\text{ext.}} = m[1-\cos(2\pi\frac{e'}{q})]\Sigma(b-a). \tag{51}$$

b and a are expressed in terms of x coordinate. Although the energy is not linear in terms of charge separation distance $d = \int_a^b dx/\sqrt{1+x^2/\lambda^2}$, but for $d \to \infty$ we have $E \to \infty$, hence the system is in confining phase.

4 Energy of finite separated charges

If we consider (45) for $m = 0$, it gives $E_{\text{ext.}} = 0$, which is not right as it does not contain the finite separation contributions. In this section we want to study the case in which the external charges have finite separation, although the energy (45), is yet the dominant long range term. In this case, the function $\tilde{\eta}$ can not be considered as a constant in the whole of the space. As has been mentioned in section one, in flat space–time the energy expression, beside the confining term, consists also of correction terms which are due to the screening behavior of the system. Here we want to calculate these correction terms for a static curved space–time.

Let us consider the general expression (36), in which we must now treat $\tilde{\eta}$ as a x-dependent expression. Before any calculation, we must say something about $G(\mathbf{x},\mathbf{x})$ in (36). As was pointed earlier, we don't know the explicit form of this function for a general curved space–time. So we will restrict ourselves to a specific condition for the metric (or the gauge coupling). We first note that on the conformally flat space–time (1), $G_\mu(\mathbf{x},\mathbf{x}')$ satisfies

$$(\frac{d^2}{dx^2} + \frac{d^2}{dt^2} - \mu^2\sqrt{g})G_\mu(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x},\mathbf{x}'). \tag{52}$$

Using the expansion

$$G_\mu(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\int G_\mu(k,x,x')e^{ik(t-t')}dk, \tag{53}$$

we obtain the following equation for $G_\mu(k,\mathbf{x},\mathbf{x}')$

$$[\frac{d^2}{dx^2} - (k^2 + \mu^2\sqrt{g})]G_\mu(k,x.x') = -\delta(x,x'). \tag{54}$$

Now if we consider the large gauge coupling regime, such that

$$\frac{dg^{\frac{1}{4}}}{dx} \ll \mu\sqrt{g}, \tag{55}$$

then the solution of (54) (at zeroth order of WKB approximation) is

$$G_\mu(k,x,x') = \frac{1}{2(k^2+\mu^2\sqrt{g(x)})^{\frac{1}{4}}(k^2+\mu^2\sqrt{g(x')})^{\frac{1}{4}}} \exp[-|\int_x^{x'} \sqrt{k^2+\mu^2\sqrt{g(u)}}du|]. \quad (56)$$

In the limit $\mathbf{x}\to\mathbf{x}'$, $G_\mu(\mathbf{x},\mathbf{x}')$ is found to be

$$\lim_{\mathbf{x}\to\mathbf{x}'} G_\mu(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\lim_{(x,t)\to(x',t')} K_0\{\mu^2\sqrt{g}[(x-x')^2+(t-t')^2]\}^{\frac{1}{2}}. \quad (57)$$

where $\delta s = g^{\frac{1}{4}}(x)[(x-x')^2+(t-t')^2]^{\frac{1}{2}}$ is the distance between two points $\mathbf{x}$ and $\mathbf{x}'$ when $\mathbf{x}'\to\mathbf{x}$. Now using $D(\mathbf{x},\mathbf{x}') = -(1/2\pi)\ln(\delta s)$ and by considering the behavior of K_0 for small arguments, $G(\mathbf{x},\mathbf{x})$ (introduced after eq.(31)) is found to be

$$2\pi G(\mathbf{x},\mathbf{x}) = -\{\gamma+\ln\frac{\mu g^{\frac{1}{4}}}{2}\}. \quad (58)$$

Thus eq.(31) yields

$$<\psi^\dagger\psi>_{m=0} = -\frac{q\exp(\gamma)}{2\pi^{\frac{3}{2}}}. \quad (59)$$

Note that in this approximation, the metric factor in eq.(31) is canceled out by the corresponding term in $G(\mathbf{x},\mathbf{x})$, and the final result is the same as the flat case as one expects from the results of [18].

Now let us go back to eq.(36). If we restrict ourselves to the regime (55), in which eq.(58) has been obtained, and change the variable ϕ as $\phi\to\phi-\eta$, the effective Lagrangian (36), for static fields with $|\phi-\eta|\ll 1$ condition, is reduced to the following Lagrangian

$$\begin{aligned} L &= \frac{1}{2}\phi(\partial_1^2-\mu^2\sqrt{g}-4\pi m\Sigma\sqrt{g})\phi - \phi(\partial_1^2-4\pi m\Sigma\sqrt{g})\eta \\ &\quad +\frac{1}{2}\eta\partial_1^2\eta - 2\pi m\Sigma\sqrt{g}\eta^2. \end{aligned} \quad (60)$$

Using (39) and (58), Σ is obtained as $\Sigma = q\exp(\gamma)/(2\pi^{3/2})$. The field equation is

$$\phi = \frac{-\partial_1^2+4\pi m\Sigma\sqrt{g}}{-\partial_1^2+\mu^2\sqrt{g}+4\pi m\Sigma\sqrt{g}}\eta. \quad (61)$$

Using the following identity which holds at large coupling limit

$$\begin{aligned} \int^x & \frac{[\sqrt{g(x)}]^\alpha}{[\mu^2\sqrt{g(x)}+4\pi m\Sigma\sqrt{g(x)}]^\beta}\exp[-\int_a^x\sqrt{\mu^2\sqrt{g(u)}+4\pi m\Sigma\sqrt{g(u)}}du]dx \\ &= -\frac{[\sqrt{g(x)}]^\alpha}{[\mu^2\sqrt{g(x)}+4\pi m\Sigma\sqrt{g(x)}]^{\beta+\frac{1}{2}}}\exp[-\int_a^x\sqrt{\mu^2\sqrt{g(x)}+4\pi m\Sigma\sqrt{g(x)}}\,dx]+c, \end{aligned} \quad (62)$$

in which c is an arbitrary constant, we obtain

$$\phi(x) = \begin{cases} \sqrt{\pi}\frac{e'}{q}\frac{\mu^2}{2}[F(x,b)-F(x,a)], & x>b, \\ \sqrt{\pi}\frac{e'}{q} - \sqrt{\pi}\frac{e'}{q}\frac{\mu^2}{2}[2F(x,x)-F(x,a)-F(x,b)], & a<x<b, \\ -\sqrt{\pi}\frac{e'}{q}\frac{\mu^2}{2}[F(x,b)-F(x,a)], & x<a, \end{cases} \tag{63}$$

where

$$F(x,y) = \frac{\sqrt{g(y)}}{f^{\frac{1}{4}}(x) f^{\frac{3}{4}}(y)} \exp[-|\int_x^y f^{\frac{1}{2}}(u)du|], \tag{64}$$

and

$$f(x) = (\mu^2 + 4\pi m\Sigma)\sqrt{g(x)}. \tag{65}$$

In deriving (63), the Green function (56) has been used. For a fix x, $F(x,y)$ is an increasing function of y in $y<x$ region and decreasing in $y>x$. So $F(x,y) \leq F(x,x)$. But $\mu^2 F(x,x) = \mu^2/(\mu^2+4\pi m\Sigma) < 1$, which is not necessarily small. Therefore the condition $|\phi-\eta| \ll 1$ is satisfied only when the factor e'/q is very small. The same arguments holds when e'/q is close to an integer number.

Putting (63) back into the Lagrangian (60) gives

$$E_{\text{ext.}} = \frac{\mu^2}{2}\{\int_a^b [\sqrt{g}\eta^2 - \mu^2\sqrt{g}\eta \frac{1}{-\partial_1^2 + \mu^2\sqrt{g} + 4\pi m\Sigma\sqrt{g}}\sqrt{g}\eta]dx\}. \tag{66}$$

Using (62) we arrive at

$$\begin{aligned} E_{\text{ext.}} &= \pi(\frac{e'}{q})^2\{(1-\frac{\mu^2}{\mu^2+4\pi m\Sigma})\frac{\mu^2}{2}\int_a^b \sqrt{g(x)}\,dx \\ &\quad + \frac{\mu^4}{4}[\frac{g(a)}{f^{\frac{3}{2}}(a)} + \frac{g(b)}{f^{\frac{3}{2}}(b)} - \frac{2\sqrt{g(a)g(b)}}{(f(a)f(b))^{\frac{3}{4}}}\exp(-\int_a^b f^{\frac{1}{2}}(u)du)]\}. \end{aligned} \tag{67}$$

The energy of external charges is composed of two parts: The first part is proportional to $\int_a^b \Sigma\sqrt{g(x)}\,dx$ and when $m\Sigma \ll \mu^2$, coincides with the confining term (45) (in the limit $e' \ll q$). On the flat space–time the remaining terms are due to the screening of external charges by dynamical fermions. On the curved space, the problem is more complicated and we can have a confining situation even in the massless Schwinger model. To see this, assume that $m=0$. The energy of external charges is then

$$E_{\text{ext.}} = \frac{e'^2}{4\mu}[g^{\frac{1}{4}}(a) + g^{\frac{1}{4}}(b) - 2g^{\frac{1}{8}}(a)g^{\frac{1}{8}}(b)\exp(-\mu\int_a^b g^{\frac{1}{4}}(u)\,du)], \tag{68}$$

which for largely separated external charges, in contrast to the flat case, does not tend to a constant, and depends on the value of the metric at the position of external charges. The condition of validity of the eq.(68) is only (55). In other words, in $m=0$ there is no need to condition $(e'/q) \ll 1$ (or $|\phi-\eta| \ll 1$) in deriving (68). For example in the massless Schwinger

model on de Sitter space (48), the energy of largely separated charges (in the coordinate x) is

$$E_{\mathrm{ext.}} = \frac{e'^2}{4\mu}[\sqrt{1+\frac{a^2}{\lambda^2}} + \sqrt{1+\frac{b^2}{\lambda^2}}]. \tag{69}$$

Although this energy is not linear in terms of charge separation distance, but for $d \to \infty$, E is infinite and the system is in confining phase.

From eq.(68), we can conclude that in contrast to the flat case, in which the energy of external charges is only a linear function of charge separation distance, on the curved space–time the energy depends also on the position of external charges. As it is clear, the infinity of the energy in a confining situation is related to the separation of charges, but the increasing rate of the energy with distance is not unique in all regions. For example in the region $x \simeq x_{\mathrm{sing}}$, where $x_{\mathrm{sing.}} \in R \bigcup \pm\infty$ is a point at which the metric is singular, two external charges located close together can have a finite energy but by moving one of the charges, the energy increases very rapidly.

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